

18F08.

Lecture 6.

Almost mathematics II.

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Setup. R, I

I flat,

$I^2 = I$.

Main examples.

① $(K, 1 \cdot 1)$ non-archimedean valued field sit.

$R = K^{\text{int}}$ = valuation ring,

$I = K^{\text{m}} = \text{maximal ideal}$.

② R perfect, $f \in R$, $I = (f^{1/p^\infty}) = \sqrt{f}$.

③ $I = \mathbb{R}$ almost = ordinary commutative algebra.

Aim of almost math: systematically ignore I -torsion.

Def. An elt $x \in M$ is almost zero if $I \cdot x = 0$.

M is almost zero if $IM = 0$.

Exercise. M is almost zero iff $I \otimes_R M = 0$.

Def. $M \xrightarrow{\varphi} N$ is an almost iso if
the ker and cokernel are almost zero.

Ex. $0 \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \rightarrow M \rightarrow M/I \rightarrow 0$
shows that $I \otimes_R M \rightarrow M$ is an almost iso.

Proposition. $\text{Mod}_{R/I} \hookrightarrow \text{Mod}_R$ is Sone.

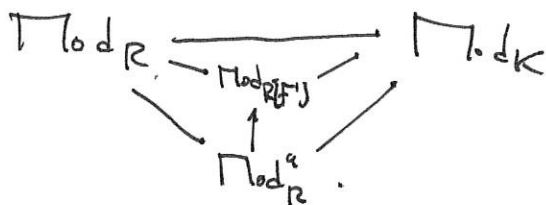
Def. The cat of almost R -modules $\boxed{\text{Mod}_R^a}$

$$\frac{\text{Mod}_R}{\text{Mod}_R/I} = \text{Mod}_R[W^{-1}]$$

↑
almost iso

$$\text{Mod}_R \xrightarrow{(-)^a} \text{Mod}_R^a \quad (\text{Bhargava: } j^+)$$

How to think of Mod_R^a (Schelec): R domain w/ fraction field K .



Think of Mod_R^a as a slightly generic fiber or as an almost integral structure.

Concrete realization. $I \otimes_R M \rightarrow M$ is an almost iso.
So, honest iso in Mod_R^a .

Let $A \subseteq \text{Mod}_R$ on M s.t. $I \otimes_R M \cong M$.

Fact: $M \in \text{Mod}_R \rightarrow I \otimes_R M \in A$.

$\left(\begin{array}{l} I \in A. \\ I \rightarrow R \text{ is an almost} \\ \text{iso.} \end{array} \right)$

Exercise. $\text{Mod}_R \xrightarrow{(-)^a} \text{Mod}_R^a$ iso to $\text{Mod}_R \xrightarrow{I \otimes_R -} A$.

Warning. A is contained in, but not equal to, the full subset of M s.t. $I \otimes M = M$.

Proposition .

$$\begin{array}{ccc} & \xleftarrow{j_! \text{ f.f. left exact}} & \\ \text{Mod } R & \xrightarrow{(-)^a} & \text{Mod } R^a \\ & \xleftarrow{j_! \text{ f.f. exact}} & \\ & \text{left adjoint} & \end{array}$$

$$j_!(M^a) = I \otimes_R M.$$

$$\text{Hom}_{R^a}(M^a, N^a) \cong \text{Hom}_R(I \otimes_R M, N).$$

↑

Do almost zero elements.

$$j_+ X = \text{Hom}_{\text{Mod } R^a}(R^a, X).$$

$(-)^a$ commutes with all limits and colimits.

$$M \in \text{Mod } R \Rightarrow j_+(M^a) = \text{Hom}_{\text{Mod } R^a}(R^a, M^a)$$

≅

$$\text{Hom}_R(I \otimes_R R, M)$$

≅

~~Proposition~~

Def. $i_+(M^a) = \text{Hom}_R(I, M) =$ almost cts of M .

Rem. On \mathcal{A} , $j_! : \mathcal{A} \hookrightarrow \text{Mod } R$.

$j_+ : \mathcal{A} \leftarrow \text{Mod } R$

$j_+(M) = \text{Hom}_R(I, M).$

$j_! \longrightarrow j_+$

$\text{Hom}_R(R, j_! H) \longrightarrow \text{Hom}_R(I, H).$

Notation. $M_! = j_!(M^0) = I \otimes_{\mathbb{R}} M$

$M_+ = j_*(M^0) = \text{Hom}_{\mathbb{R}}(I, M) \cong \text{almost cts.}$

Exercise. $M_! \xrightarrow{\text{canon}} M \xrightarrow{\text{unit}} M_+$. Both are almost isos.

Proposition/Exercise. $M \in \text{Mod}_{\mathbb{R}}$. — If M is I -torsion, $M_+ = 0 = M_!$.

— If \mathbb{R} is a domain (no zero divisors) with frac. field K ,

$$M_+ \cong \left\{ x \in M \otimes_{\mathbb{R}} K : Ix \subseteq M \right\}.$$

— If $\mathbb{R} = K^0, I = K^{\infty}$, then

$$I_+ \cong \mathbb{R},$$

$$R_+ \cong \mathbb{R}.$$

Rem. — R_+ is not always \mathbb{R} .

— \mathbb{R} perfect domain, $I = (f^{1/p^e})$,

$$R_+ \subseteq \text{Frac}(\mathbb{R}) \text{ s.t. } \forall e > 0,$$

$$f^{1/p^e} x \in \mathbb{R}.$$

Think of $f^{1/p^e} \rightarrow 1$ as $e \rightarrow \infty$, so x is almost in \mathbb{R} .