18 F05.

Lecture 6.

Almost mathematics II.

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Setup. R, I

I flat,

$I^2 = I$.

Main examples.

① $(K, |\cdot|)$ non-archimedean valued field s.t.

$R = K^{\circ}$ = valuation ring,

$I = K^{\circ\circ}$ = maximal ideal.

② R perfect, $f \in R$, $I = (f^{1/p^\infty}) = \sqrt{(f)}$.

③ $I = R$ almost = ordinary commutative algebra.

Aim of almost math: systematically ignore I-torsion.

Def. An elt $x \in M$ is almost zero if $I \cdot x = 0$.

M is almost zero if $IM = 0$.

Exercise. M is almost zero iff $I \otimes_R M = 0$.

Def. $M \xrightarrow{\varphi} N$ is an almost iso if the ker and coker are almost zero.

Ex. $0 \to \mathrm{Tor}_1(R/I, M) \to I \otimes_R M \to M \to M/I \to 0$ shows that $I \otimes_R M \to M$ is an almost iso.

Proposition. $\mathrm{Mod}_{R/I} \hookrightarrow \mathrm{Mod}_R$ is Serre.

Def. The cat of almost R-modules $\boxed{\mathrm{Mod}_R^a}$ is

$$\frac{\mathrm{Mod}_R}{\mathrm{Mod}_{R/I}} = \mathrm{Mod}_R[\underset{\substack{\uparrow \\ \text{almost isos}}}{W}{}^{-1}].$$

$$\mathrm{Mod}_R \xrightarrow[(-)^a]{} \mathrm{Mod}_R^a \qquad (\text{Bhargav}: j^*).$$

How to think of Mod_R^a (Scholze): R domain w/ fraction field K.

$$\begin{tikzcd}
\mathrm{Mod}_R \arrow[rr] \arrow[r] \arrow[dr] & \mathrm{Mod}_{R[f^{-1}]} \arrow[r] & \mathrm{Mod}_K \\
& \mathrm{Mod}_R^a \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Think of Mod_R^a as a slightly generic fiber or as an almost integral structure.

<u>Concrete realization.</u> $I \otimes_R M \to M$ is an almost iso. So, honest iso in Mod_R^a.

Let $\mathcal{A} \subseteq \mathrm{Mod}_R$ on M s.t. $I \otimes_R M \xrightarrow{\simeq} M$.

<u>Fact</u>: $M \in \mathrm{Mod}_R \Rightarrow I \otimes_R M \in \mathcal{A}$.

($I \in \mathcal{A}$. $I \to R$ is an almost iso.)

<u>Exercise</u>. $\mathrm{Mod}_R \xrightarrow[(-)^a]{} \mathrm{Mod}_R^a$ iso to $\mathrm{Mod}_R \xrightarrow[I \otimes_R -]{} \mathcal{A}$.

<u>Warning</u>. $\mathcal{A}$ is contained in, but not equal to, the full subcat of M s.t. $IM = M$.

Proposition.

$$\begin{tikzcd}[column sep=huge]
\mathrm{Mod}_R \arrow[r, "(-)^a"] & \mathrm{Mod}_R^a \arrow[l, bend right, "j_* \text{ f.f. left exact}"'] \arrow[l, bend left, "\substack{j_! \text{ f.f. exact} \\ \text{left adjoint}}"]
\end{tikzcd}$$

$$j_!(M^a) \simeq I \otimes_R M.$$

$$\mathrm{Hom}_{R^a}(M^a, N^a) \cong \mathrm{Hom}_R(I \otimes_R M, N).$$

No almost zero elements.

$$j_* X = \mathrm{Hom}_{\mathrm{Mod}_R^a}(R^a, X).$$

$(-)^a$ commutes with all limits and colimits.

$$M \in \mathrm{Mod}_R \implies j_*(M^a) = \mathrm{Hom}_{\mathrm{Mod}_R^a}(R^a, M^a) \cong \mathrm{Hom}_R(\underbrace{I \otimes_R R}_{\cong I}, M)$$

~~[illegible]~~

Def. $j_*(M^a) = \mathrm{Hom}_R(I, M) \underset{\text{def}}{=}$ almost elts of M.

Rem. On $\mathcal{A}$, $j_! : \mathcal{A} \xhookrightarrow[\text{inclusion}]{} \mathrm{Mod}_R$.

$$j_* : \mathcal{A} \longleftrightarrow \mathrm{Mod}_R$$

$$j_*(M) = \mathrm{Hom}_R(I, M).$$

$$j_! \longrightarrow j_*.$$

$$\mathrm{Hom}_R(R, j_! M) \longrightarrow \mathrm{Hom}_R(I, M).$$

Notation. $M_! = j_!(M^a) = I \otimes_R M$

$$M_* = j_*(M^a) = \mathrm{Hom}_R(I, M) = \text{almost elts.}$$

Exercise. $M_! \xrightarrow{\text{counit}} M \xrightarrow{\text{unit}} M_*$. Both are almost isos.

Proposition/Exercise. $M \in \mathrm{Mod}_R$. – If M is I-torsion, $M_* = 0 = M_!$.

– If R is a domain (more general: reduced rings) with frac. field K,

$$M_* \cong \{ x \in M \otimes_R K : Ix \subseteq M \}.$$

– If $R = K^\circ$, $I = K^{\circ\circ}$, then

$I_* \cong R$,

$R_* \cong R$.

Rem. – R_* is not always R.

– R perfect domain, $I = (f^{1/p^\infty})$,

$R_* \subseteq \mathrm{Frac}(R)$ s.t. $\forall e > 0$,

$f^{1/p^e} x \in R$.

Think of $f^{1/p^e} \to 1$ as $e \to \infty$, so x is almost in R.